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The Poincaré Group in a Demisemidirect Product with a Non-associative Algebra with Representations that Include Particles and Quarks—II

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Abstract The quarks and particles' mass and mass/spin relations are provided with coordinates in configuration space and/or momentum space by means of the marriage of ordinary Poincaré group representations with a non-associative algebra made through a demisemidirect product, in the notation of Leibniz algebras. Thus, we circumvent the restriction that the Poincaré group cannot be extended to a larger *group* by any means (including the (semi)direct product) to get even the mass relations. Finally, we will discuss a connection between the phase space representations of the Poincaré group and the phase space representations of the associated Leibniz algebra.

Keywords Leibniz algebra · Poincaré group · Particles and quarks

1 Introduction

Irreducible representations of the Poincaré group in any Hilbert space have relative position, momentum, and spin as coordinates for the various particles individually. See Sect. 4 for details. Is there any way to relate the different particles (different irreducible representations) in terms of their masses, spins, charges, etc.?

In 1965, L. O'Raifeartaigh[1] proved that the Poincaré group could not be extended to a larger *group* which would have particles as well as any relation between the various particles' masses, in spite of Okubo theory [6, 9, 10]. Moreover, a relation between the masses and spins of the particles (Regge theory) [14, 15], was similarly prohibited when you have a theory that starts with the Poincaré group. On the other hand, the quarks seemed to have nei-

[1] O'Raifeartaigh produced a paper in 1965 [11] that claimed the result. The proof was erroneous and Jost [4] and Segal [18] provided mathematical proofs. Then O'Raifeartaigh [12, 13] and with Bohm [1] extended the theorem.

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ther positions nor momenta in their theory; they do have operators for the third component of the isospin, I_3, and the hypercharges, Y_2 and Y_3, which as we shall see are eigenvalues of one component, e_7, of the set of octonions. As operators on the particles and quarks, these internal coordinates of collective masses, isospins, and hypercharges must be invariant under the action of the Poincaré group in its action on representations for massive, spinning particles. In Okubo's theory, the masses (or masses squared) of the particles are in linear relations. For Regge theory, the masses (squared) and the spin are in an affine relationship. In the case of the quarks, we have the charge $Q = I_3 + Y_2 + Y_3$, a linear relation. These are all properties of a vector space; so, why should we impose on these operators for these "internal coordinates" that they have any particular properties having to do with the *products* of these operators? In particular, why should we endow them with the property that their product is associative? By dropping the axiom of product associativity on the internal coordinates, we would have an extension of the idea of a representation of the Poincaré group to one which is no longer associative, an absolute requirement otherwise for a group. The recent work on Leibniz algebras by M.K. Kinyon [5], the book by G.M. Dixon [2], and the work by J-L. Loday and T. Pirashvili [7] and J.M. Lodder [8] on Leibniz algebras and cohomology has provided the input for a resolution of this interplay between what is a (not necessarily product associative) linear vector space property and the properties of the Poincaré group.

As we shall see, the Lie algebra of the Poincaré group may be enlarged to a *non-associative* algebra. A "Leibniz algebra", a "Lie rack", and a "Lie digroup", and the relation between the last two (Kinyon) are seen to be relevant. The appropriate non-associative algebras with which to extend the Lie algebra of the Poincaré group will be shown to be the (non-associative) octonions and algebras we can make from them (Dixon). We will provide a representation of the resulting Leibniz algebra(s). We stress the property that only when this Leibniz algebra reduces to an associative algebra is it possible to regain the properties of a particle as an irreducible representation of the Poincaré group in a Hilbert space.

The phase spaces (symplectic spaces) on which the Poincaré algebra operates may be algebraically characterized by doing a bit of cohomology; in the Lie algebra setting, the "kernel of the coboundary operator on the set of two-forms" provides the solution according to Guillemin and Sternberg [3]. The generalization of this to the Leibniz algebra case (a non-associative case) is given by Loday and Pirashvili and extended by Lodder. In this way, we will attribute the coordinates of position, momentum (and perhaps spin) to the quarks, and not just to the particles. Also, we will give (briefly) a formalism in which we have the masses and spins of particles in a family of particles so that the formulas of Okubo and Reggé hold.

In Sect. 2, we will give, in outline form, what a Leibniz algebra, a Lie rack, and a Lie digroup are, as well as defining the demisemidirect product. In Sect. 3 we will look at vector spaces that are made from certain non-associative algebras that include the octonions and indicate how we should introduce the Poincaré Lie algebra into them. These two sections are taken almost verbatim from [17]. In Sect. 4, we will discuss the representations and in Sect. 5, the cohomology of this Leibniz algebra.

2 Leibniz Algebras, Lie Racks, and Lie Digroups

In this section, we will take M. Kinyon's definition of a Leibniz algebra:

Definition 1 [5] A Leibniz algebra $(\mathfrak{l}, [\cdot,\cdot])$ is a vector space $\mathfrak{l}$ together with a bilinear mapping $[\cdot,\cdot] : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ satisfying the Leibniz identity

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{l}$.

Loday, Lodder, and Pirashvili take $[Y, [X, Z]]$ replaced by $-[[X, Z], Y]$.

If, in addition to (1), we have $[X, Y] = -[Y, X]$, then the Leibniz algebra, $\mathfrak{l}$, becomes a Lie algebra, denoted $\mathfrak{g}$, and the Leibniz identity becomes the Jacobi identity. In this case we denote $[\cdot, \cdot]$ by $[\cdot, \cdot]_{\mathfrak{g}}$.

Take [5]

$$S \equiv \{[X, X]; X \in \mathfrak{l}\}. \tag{2.2}$$

Then S is an ideal in $\mathfrak{l}$ and $\mathfrak{g} \equiv \mathfrak{l}/S$ is a Lie algebra.

Definition 2 [5] We define

$$ad_X(Y) \equiv [X, Y] \quad \forall X, Y \in \mathfrak{l}; \tag{2.3}$$

so,

$$\ker(ad) = \{X \in \mathfrak{l} : [X, \cdot] = 0\}. \tag{2.4}$$

Definition 3 [5] A Leibniz algebra $\mathfrak{l}$ is said to *split* over $\mathcal{E} \subset \mathfrak{l}$ if $\mathcal{E}$ is an ideal in $\mathfrak{l}$ such that $S \subset \mathcal{E} \subset \ker(ad)$ and there is a Lie subalgebra $\mathfrak{g} \subset \mathfrak{l}$ such that $\mathfrak{l} = \mathcal{E} \oplus \mathfrak{g}$, the direct sum of vector spaces. Then $\forall u, v \in \mathcal{E}$, and $X, Y \in \mathfrak{g}$, it follows that

$$[u + X, v + Y] = Xv + [X, Y]_{\mathfrak{g}}. \tag{2.5}$$

Conversely, given a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, form $\mathfrak{l} \equiv V \oplus \mathfrak{g}$ and define a bracket on $\mathfrak{l}$ by (2.5). Then $(\mathfrak{l}, [\cdot, \cdot])$ is a Leibniz algebra called the *demisemidirect product* of V and $\mathfrak{g}$. Then $S \simeq \mathfrak{g}V$ and $\ker(ad) = V \oplus \{X \in \mathfrak{g} : Xv = 0 \ \forall v \in V\} \cap Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$. Furthermore, $V \simeq V \oplus \{0\}$ is an ideal such that $S \subset V \subset \ker(ad)$ and $\mathfrak{l}/V \simeq \mathfrak{g}$; so, $\mathfrak{l}$ splits over V.

We will use this definition where we take $\mathfrak{g}$ equal to the Poincaré Lie algebra $\mathfrak{p}$, and V to be an appropriate $\mathfrak{p}$-module.

Having now extended the concept of a Lie algebra, we next extend the concept of a Lie group.

Definition 4 [5] A *Lie rack* $(Q, \circ, \mathbf{1})$ is a smooth manifold Q with bilinear operation $\circ$ and a distinguished element $\mathbf{1} \in Q$ such that (1) for all $x, y, z \in Q$, $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$, (2) $\forall a, b \in Q$, there exists a unique $x \in Q$ such that $a \circ x = b$, (3) $\mathbf{1} \circ x = x$, $x \circ \mathbf{1} = \mathbf{1}$ for all $x \in Q$, and such that (4) $\circ : Q \times Q \to Q$ is a smooth mapping.

Example 1 [5] Let G be a Lie group with $e =$ identity and V a G-module. On $Q = V \times G$, define binary operation $\circ$ by

$$(u, A) \circ (v, B) \equiv (Av, ABA^{-1}) \tag{2.6}$$

for all $u, v \in V$ and $A, B \in G$. Setting $\mathbf{1} = (0, e)$, then $(Q, \circ, \mathbf{1})$ is a (linear) Lie rack.

We have in mind G equal to the Lie group $\mathcal{P}$ (the Poincaré group) and V an appropriate $\mathcal{P}$-module.

It is convenient to have separate notations for operation from the left and right:

Definition 5 [5] A *dialgebra* $(\mathcal{A}, \vdash, \dashv)$ is a vector space $\mathcal{A}$ together with bilinear mappings $\vdash, \dashv: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $\forall x, y, z \in \mathcal{A}$,

$$x \vdash (y \dashv z) = (x \vdash y) \dashv z, \tag{2.7}$$

$$x \dashv (y \vdash z) = x \dashv (y \dashv z), \tag{2.8}$$

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z. \tag{2.9}$$

Lemma 1 [5] *Given a dialgebra* $(\mathcal{A}, \vdash, \dashv)$ *and defining a bracket by*

$$[x, y] \equiv x \vdash y - y \dashv x, \tag{2.10}$$

then $(\mathcal{A}, [\cdot, \cdot])$ *is a Leibniz algebra.*

Example 2 If $\mathcal{A} \equiv V \oplus G$, $G \subseteq \mathrm{End}(V)$, G a group, then define

$$(u, X) \vdash (v, Y) \equiv (Xv, XY), \tag{2.11}$$

$$(u, X) \dashv (v, Y) \equiv (0, XY). \tag{2.12}$$

One may verify that $(\mathcal{A}, \vdash, \dashv)$ satisfies (2.7), (2.8), and (2.9). Hence by Lemma 1, $(\mathcal{A}, [\cdot, \cdot])$ is the demisemidirect product of V with the group G. Again, take G to be $\mathcal{P}$, the Poincaré Lie group.

Definition 6 [5] A *disemigroup* $(H, \vdash, \dashv)$ is a set H with two binary operations $\vdash$ and $\dashv$ satisfying (1) $(H, \vdash)$ and $(H, \dashv)$ are semigroups and (2) $(H, \vdash, \dashv)$ is a dialgebra. A disemigroup $(H, \vdash, \dashv)$ satisfying (1) there exists $1 \in H$ such that $1 \vdash x = x \dashv 1 = x$ for all $x \in H$ and (2) $\forall x \in H$, there exists $x^{-1} \in H$ such that $x \vdash x^{-1} = x^{-1} \dashv x = 1$ is called a *digroup*. A *Lie digroup* $(G, \vdash, \dashv)$ is a smooth manifold G with $(G, \vdash, \dashv)$ a digroup such that the digroup operations $\vdash, \dashv: G \times G \to G$ and the inversion $(\cdot)^{-1} : G \to G$ are smooth mappings.

Example 3 [5] Let G be a group, and M a set on which G acts on the left. Suppose there exists a point $a \in M$ such that $ga = a$ for all $g \in G$ and suppose G acts transitively on $M \backslash \{a\}$. Then on $H \equiv M \times G$, define

$$(u, h) \vdash (v, k) = (hv, hk), \tag{2.13}$$

$$(u, h) \dashv (v, k) = (u, hk) \tag{2.14}$$

for all $u, v \in M$, $g, k \in G$. Then $(H, \vdash, \dashv)$ is a digroup and $(u, g)^{-1} = (a, g^{-1})$.

Now we make a definition that will get us the connection between Lie digroups and Lie racks:

Definition 7 [5] For x in digroup H, define $\circ$ on H by

$$x \circ y \equiv x \vdash y \dashv x^{-1}. \tag{2.15}$$

Then we have the

Lemma 2 [5] *Let* $(G, \vdash, \dashv)$ *be a Lie digroup. Consequently*, $(G, \circ, \mathbf{1})$ *is a Lie rack and thus the tangent space* T_1G *has the structure of a Leibniz algebra.*

Example 4 [5] Let G be a Lie group with identity e, V a G-module, and set $H = V \times G$. As in the previous example, define $\vdash, \dashv$ by (2.13), (2.14). Then $(H, \vdash, \dashv)$ is a (linear) Lie digroup. The distinguished unit is $(0, e)$ and the inverse of (u, A) is $(0, A^{-1})$.

We summarize with the theorem:

Theorem 1 [5] *Let* $H = V \times G$ *with* G *a Lie group and* V *a* G*-module. If* $(H, \vdash, \dashv)$ *is the* (*linear*) *Lie digroup defined by* (2.13), (2.14), *then the induced* (*linear*) *Lie rack is* $(H, \circ, \mathbf{1})$ *defined by* (2.6). *Conversely, every* (*linear*) *Lie rack is induced from a* (*linear*) *Lie digroup.*

Proof (Sketch) $(u, A) \circ (v, B) = (u, A) \vdash (v, B) \dashv (0, A^{-1}) = (Av, ABA^{-1})$ for all $u, v \in V$ and $A, B \in G$. □

Then we have

Theorem 2 [5] *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$, *let* V *be a* G*-module, and let* $(Q, \circ, \mathbf{1})$ *be the linear Lie rack defined by* (2.6), *where* $Q = V \times G$. *Then the tangent Leibniz algebra of* Q, T_1Q, *is the demisemidirect product* $\mathfrak{l} = V \oplus \mathfrak{g}$ *with bracket given by* (2.5). *Conversely, let* $\mathfrak{l}$ *be a split Leibniz algebra. Then there exists a linear Lie rack* Q *with tangent Leibniz algebra isomorphic to* $\mathfrak{l}$.

We will take G equal to the Poincaré Lie group $\mathcal{P}$ and $T_1G = \mathfrak{g}$ equal to the Poincaré Lie algebra $\mathfrak{p}$. V is a $\mathcal{P}$-module to be determined in the next section.

3 The Structures $V = \mathbb{Q}, \mathbb{O}, \mathbb{C} \times \mathbb{Q} \times \mathbb{O}$, etc.

We will obtain the Poincaré Lie algebra $\mathfrak{p}$ in terms of the Pauli spin algebra and then discuss the various choices for V as being a $\mathfrak{p}$-module.

First, the Poincaré group is $\mathbb{R}^4 \rtimes \mathcal{L}$, where $\mathcal{L}$ is the group of Lorentz transformations and $\mathbb{R}^4$ has the metric diag$(1, -1, -1, -1)$. $\mathbb{R}^4$ is the Minkowski space-time $\mathfrak{M}^4$. With the Cayley transform of $\mathbb{R}^4$, we identify this $\mathbb{R}^4$ with the set of real linear combinations of the Pauli spin matrices by

$$(t, x, y, z) \in \mathbb{R}^4 \mapsto t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3 \tag{3.1}$$

where σ_0 is the 2×2 identity matrix and $\sigma_1, \sigma_2, \sigma_3$ satisfy the general Pauli conditions $\sigma_j = \overline{\sigma_j}^T$, $\sigma_j\sigma_k = i\sigma_l$, (j, k, l) a cyclic permutation of $(1, 2, 3)$, and $\sigma_j^2 = \sigma_0$. Then we have $\det(t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3) = t^2 - x^2 - y^2 - z^2 = \|(t, x, y, z)\|^2$. This may be realized with the standard basis for the Pauli spin algebra, but we shall not need that here.

Now, we may obtain the action of the double cover of $\mathcal{L}$ to be $SL(2, \mathbb{C})$ where, for $A \in SL(2, \mathbb{C})$ and $p = t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3$,

$$A : p \mapsto A \cdot p = Ap\overline{A}^T. \tag{3.2}$$

In this way, we have a semidirect product $\mathfrak{M}^4 \rtimes SL(2,\mathbb{C})$ which henceforth we shall call $\mathcal{P}$. But the Pauli spin matrices are a basis for the 2×2 complex matrices as well. Hence, every element of $\mathcal{P}$ is in the complex span of the σ_js.

Let $\mathbb{Q}$ be the quaternions; i.e., the set $\text{span}\{1, q_1, q_2, q_3\}$ where $q_j^2 = -1$, and $q_j q_{j+1} = q_{j+2}$, $j = 1, 2, 3 \bmod 3$. The multiplication table for the qs is

$$\begin{array}{c|ccc} 1 & q_1 & q_2 & q_3 \\ \hline q_1 & -1 & q_3 & -q_2 \\ q_2 & -q_3 & -1 & q_1 \\ q_3 & q_2 & -q_1 & -1 \end{array} \tag{3.3}$$

Just as the reals may be embedded into the complex numbers, the complex numbers may be embedded into the quaternions, but in a non-unique way: $\mathbb{C} \hookrightarrow \mathbb{Q}$, $x + iy \mapsto x1 + yq_j$ for any q_j. In a similar fashion, we may map the Pauli spin algebra into $\mathbb{Q}$ by

$$\sigma_0 \mapsto 1, \quad i\sigma_j \mapsto -q_j, \quad \text{for } j = 1, 2, 3, \tag{3.4}$$

for any given choice of the basis for the Pauli spin operators and the basis for the quaternions. Through this map, we have an action of $\mathfrak{p}$ and hence $\mathcal{P}$ on $\mathbb{Q}$.

From [2], the set of octonions, $\mathbb{O}$, is the span of the set $\{1, e_j, j = 1, \dots, 7\}$ satisfying the multiplication table

$$\begin{array}{c|ccccccc} 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline e_1 & -1 & e_6 & e_4 & -e_3 & e_7 & -e_2 & -e_5 \\ e_2 & -e_6 & -1 & e_7 & e_5 & -e_4 & e_1 & -e_3 \\ e_3 & -e_4 & -e_7 & -1 & e_1 & e_6 & -e_5 & e_2 \\ e_4 & e_3 & -e_5 & -e_1 & -1 & e_2 & e_7 & -e_6 \\ e_5 & -e_7 & e_4 & -e_6 & -e_2 & -1 & e_3 & e_1 \\ e_6 & e_2 & -e_1 & e_5 & -e_7 & -e_3 & -1 & e_4 \\ e_7 & e_5 & e_3 & -e_2 & e_6 & -e_1 & -e_4 & -1 \end{array} \tag{3.5}$$

i.e., $e_a e_{a+1} = e_{a+5} = e_{a-2}$, $a = 1, \dots, 7 \bmod 7$. You may check that the octonion multiplication is not (always) associative. Check, for example, the multiplication of e_1, e_3 and e_5.

Now

$$\{1 \mapsto 1, q_1 \mapsto e_a, q_2 \mapsto e_{a+1}, q_3 \mapsto e_{a+5}\} \tag{3.6}$$

for any $a \in \{1, \dots, 7\} \bmod 7$ defines an inclusion $\mathbb{Q} \hookrightarrow \mathbb{O}$. Without loss of generality, choose $a = 2$. By this inclusion and (3.4), we have a natural action on the left of $\mathfrak{p}$ and hence $\mathcal{P}$ on $\mathbb{O}$.

For example, take $-i\sigma_1 \mapsto q_1 \mapsto e_2$ and then we may read the action of e_2 on the e_js from the third line of (3.5). If we set $W = \text{span}\{1, e_2, e_3, e_7\}$, then notice that $e_2 W = W$. In general, if we take $p = t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3$, then "$-ip$"$W = W$, and "$-ip$"$(O\backslash W) = O\backslash W$. But p contains the (t, x, y, z) that have a physical interpretation.

We have, in fact, a natural action of $\mathfrak{p}$ and $\mathcal{P}$ on $\mathbb{C} \otimes \mathbb{Q}$, on $\mathbb{C} \otimes \mathbb{Q} \otimes \mathbb{O} \equiv \mathbb{T}$, on $\mathbb{T}^2$, etc. Let V of the previous section equal any of these vector spaces. These are also algebras. Let V_L (V_R) equal V acting on the left (right) of V. Being algebras of operation, V_L (V_R) are associative. V also hosts a natural action of $\mathfrak{p}$ and $\mathcal{P}$ in V_L (V_R).

4 Leibniz Representations

We have a generally non-associative algebra, V, on which the Poincaré group acts. We must make up a Leibniz algebra from the two. Now, in [7] we find the definition of a representation of a Leibniz algebra to be

Definition 8 A representation of the Leibniz algebra $\mathfrak{l}$ is a k-module M equipped with two actions (left and right) of $\mathfrak{l}$, $[-,-]:\mathfrak{l}\times M\to M$ and $[-,-]:M\times\mathfrak{l}\to M$ satisfying

$$[m,[x,y]]=[[m,x],y]-[[m,y],x], \tag{4.1}$$

$$[x,[m,y]]=[[x,m],y]-[[x,y],m], \tag{4.2}$$

$$[x,[y,m]]=[[x,y],m]-[[x,m],y], \tag{4.3}$$

for all x and $y\in\mathfrak{l}$ and $m\in M$.

Clearly, the last two axioms are equivalent when $[x,[m,y]]+[x,[y,m]]=0$. The condition $[x,m]=-[m,x]$ implies the equivalence of all three and is the condition for an ordinary Lie algebra representation. We will keep to the common convention that we have the right action on the internal coordinates.

In $\mathbb{O}_R$, define $I_{a+2}\equiv e_{Ra+5,a+1,a}$, where $e_{Rabc}(m)\equiv((me_a)e_b)e_c$. These I_{a+2} are all diagonal matrices with $I_{a+2}(e_b)=-e_b$, $b=a+2,a+3,a+4,a+6$ and $I_{a+2}(e_b)=+e_b$ otherwise. (This is addition mod 7.) For example, let us take $a=1$ and compute

$$I_3(e_4)=((e_4e_6)e_2)e_1=(e_7e_2)e_1=e_3e_1=-e_4. \tag{4.4}$$

Similarly, we define e_{Labc} by $e_{Labc}(m)=e_a(e_b(e_cm))$ and compute $e_{La+5,a+1,a}=e_{Ra+5,a+1,a}$; so, everything we will say about $I_{a+2}\in\mathbb{O}_R$ holds equally for $I_{a+2}\in\mathbb{O}_L$. In addition, we compute that

$$e_{Ra\cdots bc\cdots d}=-e_{Ra\cdots cb\cdots d}\quad\text{for } b\neq c, \tag{4.5}$$

and

$$e_{Rab\cdots pp\cdots c}=-e_{Rab\cdots c}. \tag{4.6}$$

From this we obtain the result $(e_{Rabc})^2=e_{Rabcabc}=1_R$ for a,b,c all different. Furthermore,

$$\frac{1}{2}(1_R+e_{Rabc})\quad\text{and}\quad\frac{1}{2}(1_R-e_{Rabc}) \tag{4.7}$$

are orthonormal idempotents for $\mathbb{O}_R$ (which equals $\mathbb{O}_L$). These are all results of Dixon [2, pp. 38–39, 43–45].

First, we will try for M just $\mathbb{O}$. After a fair amount of computation we find that $\mathbb{O}=M$ with $[m,x]=(e_{R732}m)x=(e_{L732}m)x$ will satisfy (4.1) (with the choice for (3.6) of $a=2$) for $\mathfrak{g}=\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{C})$, the Lie algebras for Lie groups $SU(2)$ and $SL(2,\mathbb{C})$, respectively. (We may choose any $a\in\{1,\ldots\}$ for (3.6) and the choice of I_{a+2}.) But this $\mathfrak{g}$ is a Lie algebra; so, we must have $[x,m]=-[m,x]$ in order to have the conditions (4.1), (4.2), and (4.3) satisfied. We remark that here we obtain again an ordinary representation on the set spanned by $1,e_a,e_{a+1},e_{a+5}$, and a peculiar "twist" for the rest.

There is a second way [2, pp. 46–49] that we might obtain a Lie group when operating on $\mathbb{O}$. The set

$$\{e_{Lab} - e_{Lcd} : e_a e_b = e_c e_d\} \tag{4.8}$$

forms a basis for the Lie algebra for the 14 dimensional exceptional Lie group G_2. In fact G_2 is the automorphism group of $\mathbb{O}$, acting on the left. Let us denote it by LG_2. Acting on the right we have

$$\{e_{Rab} - e_{Rcd} : e_a e_b = e_c e_d\}, \tag{4.9}$$

since $e_{Lab} - e_{Lcd} = -(e_{Rab} - e_{Rcd})$ when $e_a e_b = e_c e_d$. In general, $e_{Rab} - e_{Rcd}$ generates actions that leave 1 and e_f invariant for $f \neq a, b, c, d$.

For example, let $\mathbb{Q} \hookrightarrow \mathbb{O}$, $q_1 \mapsto e_2, q_2 \mapsto e_3, q_3 \mapsto e_7$. Then $e_{Rab} - e_{Rcd} \in LG_2$ with (a, b, c, d) equal to a permutation of $(1, 4, 5, 6)$ generates the following Lie algebra elements which leave $1, e_2, e_3, e_7$ invariant. There are three choices: $a = 1$ and $b \in \{4, 5, 6\}$, corresponding to $e_a e_b = -e_3, e_7, -e_2$. Now, let

$$A = e_{R14} - e_{R65}, \qquad B = -e_{R15} + e_{R46}, \qquad C = e_{R16} - e_{R54}. \tag{4.10}$$

From (4.4) and (4.5), we obtain $[A, B] = 4C$, etc.; i.e., $A/2$, $B/2$, $C/2$, generate the Poincaré algebra for the Poincaré group which we shall denote by $\mathfrak{p}_0$ resp. $\mathcal{P}_0$. Thus we obtain a representation of $\mathcal{P}_0$ on $\mathbb{O}$. So we have $\mathcal{P}_0 \subset LG_2$. This $\mathcal{P}_0$ leaves e_2, e_3, e_7 invariant and so will not provide e_2, e_3, e_7 with the ps and qs coming from the Poincaré algebra as below, but will do so for the remaining four e_fs.

What is the connection between these realizations of the Poincaré group? We will take the $a = 2$ representation of $\mathbb{Q}$ in $\mathbb{O}$: $q_1 \mapsto e_2, q_2 \mapsto e_3, q_3 \mapsto e_7$. Let $\operatorname{Im}(\mathfrak{p})$ denote the result of this map. As we have seen, the inclusion $\mathfrak{p} \hookrightarrow \mathbb{O}_R$ given by $x \mapsto e_{R732} \operatorname{Im}(x)$ will give the representation of $\mathfrak{p}$ as a Lie algebra, which we will call $\mathfrak{p}_1$. The map $x \mapsto e_{R732} \frac{1}{2}(1_R + e_{R732}) \operatorname{Im}(x) = \frac{1}{2}(1_R + e_{R732}) \operatorname{Im}(x)$ gives us a second representation of $\mathfrak{p}$ as a Lie algebra, which we denote by $\mathfrak{p}_2$. It is the "usual" representation of $\mathfrak{p}$ on $\operatorname{Im}(\mathbb{Q})$. The map $x \mapsto e_{R732} \frac{1}{2}(1_R - e_{R732}) \operatorname{Im}(x) = -\frac{1}{2}(1_R - e_{R732}) \operatorname{Im}(x)$ gives us a third representation of $\mathfrak{p}$ as a Lie algebra, which we have previously denote by $\mathfrak{p}_0$. Clearly $\mathfrak{p}_1 = \mathfrak{p}_2 + \mathfrak{p}_0$.

Now there is a symmetry about the "third" axis of $\mathfrak{M}^4$ for the Poincaré group in the case of massive, spinning particles at the origin of phase space. Specifically, one has the following: Representations (resp. phase space representations) of the massive spinning case of $\mathcal{P} = \mathbb{R}^4 \rtimes \mathcal{L} = \mathbb{R}^4 \rtimes SL(2, \mathbb{C})$ are formed [16] by taking the subgroup $H = \mathcal{L}_{p_*, s_*} = [SL(2, C)]_{p_*, s_*}$ (resp. $H' = \mathbb{R}p_* \rtimes \mathcal{L}_{p_*, s_*} = \mathbb{R}p_* \rtimes [SL(2, C)]_{p_*, s_*}$) where $p_* = m(1, 0, 0, 0) \in \mathbb{R}^4$ is the momentum at rest, $s_* = S(0, 0, 0, 1)$ is the spin at rest, m is the mass, S is the spin, and where $\mathcal{L}_{p_*, s_*}$ is the part of $\mathcal{L}$ that fixes p_* and s_*. We note that $p_*^\mu (s_*)_\mu = 0$, a reflection of the property that the magnetic moment is orthogonal to the momentum. The representation spaces are then $\mathcal{P}/H$ (resp. $\mathcal{P}/H'$). We remark that we have $\mathcal{P}$-invariance of these spaces and that $\mathcal{L}/\mathcal{L}_{p,s} = SU(2)$. We also have, for $A \in SU(2)$, that (1) $A \cdot p_* = A p_* A^\dagger = p$ for some $p \in \mathfrak{M}^4$ and (2) $A \cdot s_* = A s_* A^\dagger = s$ for some $s \in \mathfrak{M}^4$. Then we have $p_\mu p^\mu = m^2$, $s_\mu s^\mu = -S^2$, and $p^\mu s_\mu = 0$; i.e. we have the spin and momentum coordinates. There is in fact one element A that satisfies (1) and (2). We will call that element $A_{p,s}$. In this fashion, $SL(2, \mathbb{C})/[SL(2, \mathbb{C})]_{p_*, s_*} \approx \mathbb{R}^3_{\text{momentum}} \rtimes \mathfrak{S}^2$ where the spin space is $\mathfrak{S}^2(p) = \{s \in \mathfrak{M}^4 : p \cdot s = 0, s \cdot s = -S^2\}$. Then $(q, A) = (q', A')(\lambda p_*, B)$ for $(\lambda p_*, B) \in H'$ iff $q = q' + \lambda p' = q' + \lambda A_{ps} \cdot p_*$. We make the choice $\lambda = -m^{-2}(q \cdot p)$. Summarizing, we have $\pi : \mathcal{P} \to \mathcal{P}/H'$, $\pi : (q, A) \mapsto (q + \mathbb{R}p, A_{p,s} SL(2, \mathbb{C})_{p_* s_*}) \equiv [(q, p, s)]$. So, the Poincaré group has the coordinates (q, p, s) built into it in the representation for

$m \neq 0$ and $S \neq 0$. Note that you have the spin in the 3-direction of $\mathfrak{M}^4$ only if A_{ps} leaves the 3-axis invariant (so $s = s_*$). $\mathcal{H}$ is the space of complex-valued functions that are square integrable over $\mathcal{P}/H'$ with respect to the invariant measure, or one of the irreducible spaces that you get from this $\mathcal{H}$.

In view of Theorem 2, $\mathcal{H} \times \mathcal{P}$ is a linear Lie rack with Leibniz algebra $\mathcal{H} \oplus \mathfrak{p}$. We may generalize this to $\mathfrak{l} = (\mathcal{H} \times V) \oplus \mathfrak{p}$ with V as in the last paragraph of the last section. Alternatively, we may take $M = \mathbb{O}$, or for that matter $M = V$, and $\mathfrak{l} = \mathcal{H} \oplus \mathfrak{p}$. In the latter case, $[m, x]$ is the appropriate tensor product operator of $(mI_{a+2})x$ whenever $\mathbb{O}$ is present and mx otherwise. Then, the action of $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathfrak{C})$ (i.e., of $\mathbb{Q}$) on V decomposes into two parts: the part containing only (the injection of) $\mathbb{Q}$, and the part containing "$V \backslash \mathbb{Q}$".

Dixon [2] made the discovery that we may write the isospin, the hypercharges, and the charge operators in terms of the LG_2-group actions on $V = \mathbb{T}^2$. Dixon shows that the operators (in the theory of quarks) are the third component of the isospin $\widehat{I}$, I_3, in the $\mathbb{C} \times \mathbb{Q}$ part of $\mathbb{T}$; the hypercharges $Y_2 = -e_7/2 \in \mathbb{T}^2_R$ and Y_3 a linear combination of $-ie_7/6 \in \mathbb{T}^2_L$ and $ie_7/6 \in \mathbb{T}^2_R$. So, we look for Poincaré transformations which leave q_3 and e_7 invariant. Now we may form the stability subgroup of LG_2 stabilizing 1 and e_7 with basis $\{e_{Rab} - e_{Rcd} : e_a e_b = e_c e_d, a, b, c, d \neq 7\}$. Then we obtain an $SU(3)$ subgroup of LG_2. This $SU(3)$ equals the color group for quarks! Clearly $\mathcal{P}_0 \subset SU(3)$ for the $\mathcal{P}_0$ just constructed. When working on $\mathbb{T}^2$, we also have the orthogonal group in one variable, $O(1)$; so, we may talk about the $O(1)$, $\mathcal{P}_0$, and $SU(3)$ representations there. Other groups are generated in a similar fashion, and play a role in "weak mixing", gauge fields, etc. We will not derive them here, but they are derived in a similar fashion as those of $O(1)$, $SU(2)$, $SU(3)$.

Next, we will work with the Leibniz algebra $\{(t, g); t \in \mathcal{H} \times \mathbb{T}^2, g \in \mathfrak{p}\}$ as obtained on $V = \mathcal{H} \times \mathbb{T}^2$ instead of on $\mathbb{T}^2$. What remains to be checked is that the various operators that Dixon and his forerunners have used are Poincaré invariants for one of the $\mathfrak{p}_i$. Then, since $\widehat{p}_\mu \widehat{p}^\mu = m^2$ and (the isospin) $\widehat{I}_\mu \widehat{I}^\mu = I(I + 1)$ are Poincaré invariants, we may obtain the Okubo mass formula [6, 9, 10] in any representation of $\mathcal{P}$. In the representation $\mathcal{P}_1$, the operators that Nixon and his forerunners have used in classifying the particles and quarks are just at s_* only, which singles out the "3-axis" (q_3 and e_7)! The Poincaré transformations on $\mathcal{P}/H'$ then may be used to express these operators at general (q, p, s). In the representation, $\mathcal{P}_0$, we may use the Hilbert space for any mass/spin as only the fact that e_7 is invariant is used. Thus, working in the demisemidirect product of V with $\mathcal{P}$, then, you have in addition the "position" and "momentum", suitably generalized, for any of these objects. It remains to be seen which of these representations fits the physics further. But we are not finished with quark theory yet; we have to have the property that certain triplets of quarks give rise to the particles. A particle is also something in $(\mathcal{H} \times \mathbb{T}^2) \oplus \mathfrak{p}$ that has none of the general properties of $\mathbb{T}^2$ that deal with the quarks proper. In particular, we must assign the quarks the labels of e_1, e_4, e_5, e_6 so that the sum of the corresponding three of the coefficients of e_j, $j \in \{1, 4, , 5, 6\}$ vanish when added, as that is what a particle would have. Nixon's results have precisely that, although it is obscure in his notation. Also, they would have to have the $\mathcal{H}$ part of the "wave function" all the same so that when you added, you would get just addition in the $\mathbb{T}^2$ part. This leaves the span of e_j, $j \in \{2, 3, 7\}$, which form a representation of $\mathfrak{su}(2)$, the spin, and so is allowed for a particle. This is a duplication, as the spin is also represented in $\mathcal{H}$. (But, maybe the representation here is not the spin but an internal coordinate.???) The only other route to particles is to have all coefficients of the e_j add to zero which puts constraints on the coefficients. Then $\mathbb{T}^2$ would reduce to $\mathbb{C}$, and you would obtain the particles as just being in $\mathcal{H}$, as they should, but then the coefficient of $e_7 = 0$.

If we obtained the "other groups" in this way, then we would have the symmetry with which we would derive the mass formulas and the mass/spin formulas for the particles. We shall not do that here.

We have achieved models in which we may attribute momentum, position, and spin to the quarks, as well as the particles, and in which we may discuss the $\mathcal{P}$-invariance of the operators which are inputs for the mass formulas and the mass/spin relations.

5 (Co)homology

We are left with the question "How does the phase space $\mathcal{P}/H'$ arise, and does that derivation have an equivalent in the Leibniz algebra set up?" Guillemin and Sternberg [3] have shown that for any Lie algebra/Lie group, the phase spaces (symplectic spaces) on which the group may act are all found in a certain set determined by the coboundry operator. The coboundary operator is something entering into Lie (co)homology. $\mathcal{P}/H'$ is just one example of one such phase space [16]. Now

$$d(g_1 \otimes \cdots \otimes g_n) = \sum_{1 \le i < j \le n} (-1)^j g_1 \otimes \cdots \otimes g_{i-1} \otimes [g_i, g_j] \otimes g_{i+1} \otimes \cdots \otimes \widehat{g_j} \otimes \cdots \otimes g_n, \quad (5.1)$$

where the $g_i \in$ Lie algebra $\mathfrak{g}$, defines the (Lie) boundary map. One may use the analog in the Leibniz algebra setting [7]! If all the (co)homology theory and the identification of the symplectic spaces remains the same in analog, then the picture for the demisemidirect product of V with $\mathfrak{p}$ would be complete. Although specific examples have been worked out, there is as yet no general theory that will do the trick although there has been some progress [7, 8]. This is a point at which we will just say that it is under consideration.

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